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# Harmonic oscillator pre-potentials in $SU(2)$ lattice gauge theory

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## Abstract

We write the  $SU(2)$  lattice gauge theory Hamiltonian in  $d$  dimensions in terms of pre-potentials which are  $SU(2)$  fundamental doublets of harmonic oscillators. The Hamiltonian in terms of pre-potentials has  $SU(2) \otimes U(1)$  local gauge invariance. These pre-potentials enable us to solve the  $SU(2)$  Gauss law and characterize the  $SU(2)$  gauge invariant Hilbert space in terms of a set of integers. We discuss the consequences of the additional  $U(1)$  gauge invariance. The extension to  $SU(N)$  lattice gauge theory is discussed.

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## 1. Introduction

Gauge theories form the underlying framework for both strong and electroweak interactions. Quantizing them on a lattice [1] provides a way to go beyond perturbative expansions. As the non-perturbative issues of the lattice gauge theories, like colour confinement, are still not understood it is useful and desirable to work with different possible formulations of lattice gauge theories. To date most of the work in lattice gauge theories has been done using the action formulation. The advantage of this approach is that it readily lends itself to Monte Carlo simulations. However, the alternative Hamiltonian framework [2] has its own advantages as it brings in a different intuition and insight into the problem. For example, the glueball or hadron masses come directly from the eigenvalues of the Hamiltonian in the physical Hilbert space [3] and are not extracted from the correlation functions as done in Monte Carlo simulations. Further, the Hamiltonian approach has been useful in the context of redefinition of action [4], the Wilson–Polyakov confinement test [5] and to study the vacuum structure [6]. The Hamiltonian framework is also used to develop non-perturbative techniques, such as strong coupling expansions [3], variational methods [7],  $t$ -expansions [8], plaquette expansion [9] and coupled cluster method [10, 11]. Monte Carlo techniques have been developed to study the spectrum of the lattice gauge theory Hamiltonian [12]. In the recent past, the improved

discretization programme has also been applied to the Hamiltonian formulation [13]. In all the above approaches, the Hamiltonian formulation used is that of Kogut and Susskind [2]. This formulation involves group valued link operators and their conjugate variables which are the electric fields belonging to the adjoint representation of the gauge group (section 2). In this work, using the Jordan–Schwinger map [14, 15], we propose a reformulation of the Kogut–Susskind Hamiltonian in terms of pre-potentials defined on the links of the lattice. These pre-potentials are harmonic oscillators and belong to the fundamental representations of the gauge group. We show that the  $SU(N)$  lattice gauge theory written in terms of pre-potentials has  $SU(N) \otimes U(1)^{N-1}$  gauge invariance. Thus the pre-potential formulation provides an alternative way of looking at and analysing lattice gauge theories in terms of simple harmonic oscillators. A further advantage of the pre-potential formulation is that under the non-Abelian gauge transformations, the harmonic oscillators or pre-potentials transform according to the fundamental (*not adjoint*) representations of the gauge group (section 4). This simplification in the gauge transformation properties, as expected, leads to an alternative way of solving the Gauss law constraints. We explicitly construct a manifestly gauge invariant basis in terms of harmonic oscillators. This basis is labelled by a set of gauge invariant integers at every lattice site (section 5). We compare our construction with the standard basis generated by the set of all Wilson loops [16] and the basis given in terms of Wigner  $D$  matrices [11, 17, 18]. In the strong coupling limit, we show that it is the Abelian gauge group of the pre-potential formulation which leads to the confining strings between two external quarks. Further, different possible gauge invariant ways of self-coupling of pre-potentials and coupling of matter with pre-potentials are discussed. We illustrate the above ideas using the simple  $SU(2)$  gauge group. The generalization to the  $SU(N)$  gauge group is discussed at the end.

The important ingredient involved in going from adjoint representation (electric field) to the fundamental representation (pre-potentials) is the use of the Jordan–Schwinger map of the  $SU(N)$  Lie algebra [14, 15]. The Jordan–Schwinger mapping converts the ( $SU(N)$ ) Lie algebras to a set of harmonic oscillator algebras [15] thus simplifying not only the algebraic structures but also the representations of the  $SU(N)$  group. This mapping is especially useful for lattice gauge theories because, unlike continuum electric fields, the lattice electric fields do not commute and follow the Lie algebra of the gauge group [2]. In fact, the Jordan–Schwinger mapping has been found useful in many different areas of physics. In quantum physics and optics, it is used to construct  $SU(2)$  and  $SU(N)$  coherent states analogous to harmonic oscillator coherent states [15, 20]. In nuclear physics, assuming nucleus to be a rigid rotator, it is convenient to describe high spin states of the nucleus [21] in terms of these oscillator states. In condensed matter physics, the  $SU(2)$  Heisenberg spin chains are analysed in terms of these oscillators [22]. Further, the study of representations of certain non-compact Lie groups in terms of Jordan–Schwinger-type harmonic oscillators has been found useful to study supersymmetry [23]. In the context of QCD formulated as a quantum link model [24], this mapping maps the operators of the theory into rishons which are anti-commuting harmonic oscillators.

The plan of the paper is as follows. In section 2, we start with an introduction to  $SU(2)$  lattice gauge theory in Hamiltonian formulation. This section is for the sake of completeness and setting up the notation. For more details, the reader is referred to [2, 17]. In section 3, we describe the Jordan–Schwinger mapping, the pre-potential operators and the associated  $U(1)$  gauge invariance. We must emphasize that this Abelian gauge invariance is *not a subgroup* of  $SU(2)$  and is defined on the links. In section 4, we study the  $SU(2)$  gauge transformation properties of the pre-potentials. Section 5 is devoted to the study of physical Hilbert space in terms of pre-potential operators. At the end, we briefly mention some of the corresponding results for  $SU(N)$  lattice gauge theories.

## 2. The Hamiltonian formulation

We start with  $SU(2)$  lattice gauge theory in  $d$  dimensions. The Hamiltonian [2] is

$$H = \sum_{n,i} \sum_{a=1}^3 E^a(n,i) E^a(n,i) + K \sum_{\text{plaquettes}} \text{Tr}(U_{\text{plaquette}} + U_{\text{plaquette}}^\dagger). \quad (1)$$

where

$$U_{\text{plaquette}} = U(n,i)U(n+i,j)U^\dagger(n+j,i)U^\dagger(n,j)$$

and  $K$  is the coupling constant. The index  $n$  labels the site of a  $d$ -dimensional spatial lattice and  $i, j$  ( $= 1, 2, \dots, d$ ) denote the directions of the links. The traces in (1) are over the spin half indices of the  $SU(2)$  gauge group. Each link  $(n, i)$  is associated with a symmetric top, whose configuration (i.e., the rotation matrix from space-fixed to body-fixed frame) is given by the operator valued  $SU(2)$  matrix  $U(n, i)$ .  $E^a(n, i)$  are the  $SU(2)$  electric field operators. The quantization rules are [2, 17]

$$[E^a(n, i), U(n, i)] = \frac{\sigma^a}{2} U(n, i) \Rightarrow [E^a(n, i), E^b(n, i)] = -i\epsilon^{abc} E^c(n, i). \quad (2)$$

The second commutation relation above follows from the first as a consequence of the Jacobi identity. We note that the angular momentum operators are  $-E^a(n, i)$ . The Hamiltonian (1) and the quantization rules in (2) are invariant under  $SU(2)$  gauge transformations,

$$E(n, i) \rightarrow V(n)E(n, i)V^\dagger(n) \quad U(n, i) \rightarrow V(n)U(n, i)V^\dagger(n+i). \quad (3)$$

where  $E(n, i) \equiv E^a(n, i) \frac{\sigma^a}{2}$ . The commutation relations (2) show that the generators of left gauge transformations are  $E^a(n, i)$ . We therefore associate  $E^a(n, i)$  with the left of the link  $(n, i)$  and call them left electric fields. They can be interpreted as the angular momentum operators of the symmetric top in the body-fixed frame. To find the generators of right gauge transformations in (3), we define

$$e(n, i) \equiv e^a(n, i) \frac{\sigma^a}{2} = U^\dagger(n, i)E(n, i)U(n, i). \quad (4)$$

It is easy to check that  $e^a(n, i)$  satisfy

$$[e^a(n, i), U(n, i)] = U(n, i) \frac{\sigma^a}{2} \Rightarrow [e^a(n, i), e^b(n, i)] = i\epsilon^{abc} e^c(n, i). \quad (5)$$

The commutation relations (5) show that the generators of right gauge transformations on  $U(n, i)$  are  $e^a(n, i)$ . The defining equation (4) shows that under  $SU(2)$  gauge transformations,

$$e(n, i) \rightarrow V(n+i)e(n, i)V^\dagger(n+i). \quad (6)$$

We therefore associate  $e^a(n, i)$  with the right of the link  $(n, i)$  and call them right electric fields. Further, it is easy to check that the left and right electric fields commute amongst themselves:

$$[E^a(n, i), e^b(n, i)] = 0. \quad (7)$$

Therefore,  $e^a(n, i)$  can be interpreted as the angular momentum operators of the symmetric top in the space-fixed frame of the symmetrical top. By construction, they satisfy the kinematical constraints,

$$\sum_{a=1}^3 e^a(n, i)e^a(n, i) = \sum_{a=1}^3 E^a(n, i)E^a(n, i), \quad (8)$$

on each link  $(n, i)$ . From (3) and (6) the  $SU(2)$  Gauss law at site  $(n)$  is

$$\sum_{i=1}^d (E^a(n, i) - e^a(n - i, i)) = 0. \quad (9)$$

It simply states that the sum of all the  $2d$  angular momenta meeting at a site  $(n)$  is zero. We will solve this Gauss law to construct  $SU(2)$  gauge invariant Hilbert space in terms of pre-potentials in section 5.

### 3. The pre-potentials and Abelian gauge invariance

We consider two independent doublets of harmonic oscillators,  $(a_\alpha, a_\alpha^\dagger)$  and  $(b_\alpha, b_\alpha^\dagger)$  on each link  $(n, i)$ . They satisfy

$$[a_\alpha, a_\beta^\dagger] = \delta_{\alpha, \beta}, \quad [b_\alpha, b_\beta^\dagger] = \delta_{\alpha, \beta} \quad \alpha, \beta = 1, 2. \quad (10)$$

Using the Jordan–Schwinger boson representation of  $SU(2)$  Lie algebra [14], we write

$$E^a(n, i) \equiv a^\dagger(n, i) \frac{\tilde{\sigma}^a}{2} a(n, i), \quad e^a(n, i) \equiv b^\dagger(n, i) \frac{\sigma^a}{2} b(n, i). \quad (11)$$

Note that in (11),  $\tilde{\sigma}_{\alpha\beta}^a \equiv \sigma_{\beta\alpha}^a$  is used to get the negative sign on the rhs of equation (2). More explicitly on each link,

$$\begin{aligned} E^1 &= \frac{1}{2}(a_2^\dagger a_1 + a_1^\dagger a_2), & E^2 &= \frac{-i}{2}(a_2^\dagger a_1 - a_1^\dagger a_2), & E^3 &= \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2) \\ e^1 &= \frac{1}{2}(b_2^\dagger b_1 + b_1^\dagger b_2), & e^2 &= \frac{i}{2}(b_2^\dagger b_1 - b_1^\dagger b_2), & e^3 &= \frac{1}{2}(b_1^\dagger b_1 - b_2^\dagger b_2). \end{aligned} \quad (12)$$

The two Casimirs of left and right gauge rotations on every link are given by

$$\sum_{a=1}^3 E^a E^a = \frac{a^\dagger \cdot a}{2} \left( \frac{a^\dagger \cdot a}{2} + 1 \right), \quad \sum_{a=1}^3 e^a e^a = \frac{b^\dagger \cdot b}{2} \left( \frac{b^\dagger \cdot b}{2} + 1 \right). \quad (13)$$

In (13),  $a^\dagger \cdot a = a_1^\dagger a_1 + a_2^\dagger a_2$  and  $b^\dagger \cdot b = b_1^\dagger b_1 + b_2^\dagger b_2$  are the total number operators corresponding to  $a$  and  $b$  type oscillators. Thus the representation (11) allows us to describe the left and right electric fields in terms of harmonic oscillators. In the next section, we show that it is also simple to express the link operators  $U(n, i)$  in terms of these oscillators. Therefore, we also call them pre-potentials. We note that the defining equations (11) imply  $U(1) \otimes U(1)$  gauge invariance on every link:

$$a_\alpha^\dagger(n, i) \rightarrow \exp(i\theta(n, i)) a_\alpha^\dagger(n, i), \quad b_\alpha^\dagger(n, i) \rightarrow \exp(i\phi(n, i)) b_\alpha^\dagger(n, i). \quad (14)$$

In (14),  $\theta(n, i)$  and  $\phi(n, i)$  are the arbitrary phase angles at each link  $(n, i)$ . Note that this  $U(1) \otimes U(1)$  group is *not a subgroup* of the  $SU(2)$  gauge group. However, constraint (8) and relations (13) imply that the occupation numbers of the  $a$  and  $b$  type harmonic oscillator pre-potential on each link are equal; i.e.,

$$\sum_{\alpha=1}^2 a_\alpha^\dagger(n, i) a_\alpha(n, i) = \sum_{\alpha=1}^2 b_\alpha^\dagger(n, i) b_\alpha(n, i) \equiv N(n, i). \quad (15)$$

Therefore,  $\tilde{\mathcal{H}}$  of pure  $SU(2)$  lattice gauge theory is characterized by the following orthonormal state vectors at each link:

$$\begin{bmatrix} n & N - n \\ \bar{n} & N - \bar{n} \end{bmatrix} \equiv \frac{(a_1^\dagger)^n (a_2^\dagger)^{N-n} (b_1^\dagger)^{\bar{n}} (b_2^\dagger)^{N-\bar{n}}}{\sqrt{n!} \sqrt{(N-n)!} \sqrt{\bar{n}!} \sqrt{(N-\bar{n})!}} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \quad (16)$$

The invariance of  $\tilde{\mathcal{H}}$  under (14) implies that the gauge group  $U(1) \otimes U(1)$  reduces to  $U(1)$  with  $\theta(n, i) = -\phi(n, i)$ . The constraint (15) now becomes the Gauss law for this resulting Abelian gauge invariance. We note that the standard construction of  $\tilde{\mathcal{H}}$  on a link involves link operators,

$$|j, m, \bar{m}\rangle = \sum_{i_1, i_2, \dots, i_{2j} \in S_{2j}} U_{m_{i_1} \bar{m}_1} U_{m_{i_2} \bar{m}_2} \cdots U_{m_{i_{2j}} \bar{m}_{2j}} |0\rangle \quad (17)$$

where  $j, m (= m_{i_1} + m_{i_2} + \cdots + m_{i_{2j}})$  and  $\bar{m} (= \bar{m}_1 + \bar{m}_2 + \cdots + \bar{m}_{2j})$  denote the eigenvalues of the complete set of commuting operators (CSCO) on every link  $E^a E^a, E^3$  and  $e^3$  respectively,  $|0\rangle$  is the vacuum satisfying  $E^a |0\rangle = e^a |0\rangle = 0$ .  $S_{2j}$  is the permutation group with  $(2j)!$  elements which act on the indices  $(i_1, i_2, \dots, i_{2j})$  to symmetrize the right-hand side of (17) in the magnetic quantum numbers. The correspondence with (16) is  $2j = N, m = 2n - \frac{N}{2}$  and  $\bar{m} = 2\bar{n} - \frac{N}{2}$ . We note that the pre-potential construction (16) of the eigenstates of CSCO is simpler than the corresponding construction (17) using the link operators  $U_{mm'}$  as it does not require the permutation group  $S_{2j}$ .

#### 4. The $SU(2)$ gauge invariance

Under the  $SU(2)$  gauge transformations (3) and (6),

$$a_\alpha^\dagger(n, i) \rightarrow V_{\alpha\beta}(n) a_\beta^\dagger(n, i), \quad b_\alpha^\dagger(n, i) \rightarrow b_\beta^\dagger(n, i) V_{\beta\alpha}^\dagger(n + i). \quad (18)$$

Thus, the two sets of pre-potentials transform like  $SU(2)$  doublets, one from the left and the other from the right. We further define that  $\tilde{a}_\alpha \equiv \epsilon_{\alpha,\beta} a_\beta$  and  $\tilde{b}_\alpha \equiv \epsilon_{\alpha,\beta} b_\beta$ . Under  $SU(2)$  gauge transformations,  $\tilde{a}_\alpha$  and  $\tilde{b}_\alpha$  transform as  $a_\alpha^\dagger(n, i)$  and  $b_\alpha^\dagger(n, i)$  respectively. Exploiting the above symmetry properties, we now directly write down the operator valued  $SU(2)$  matrix  $U(n, i)$  in the Hilbert space  $\tilde{\mathcal{H}}$  as

$$U(n, i)_{\alpha\beta} = F(n, i) (a^\dagger(n, i)_\alpha b^\dagger(n, i)_\beta + \tilde{a}(n, i)_\alpha \tilde{b}(n, i)_\beta) F(n, i). \quad (19)$$

Note that we have not included terms of type  $a_\alpha^\dagger \tilde{b}_\beta$  which, though consistent with  $SU(2)$  gauge transformations (3), are not invariant under  $U(1)$  gauge transformations (14) with  $\theta(n, i) = -\phi(n, i)$ . In (19),  $F(n, i) \equiv \frac{1}{\sqrt{N(n, i)+1}}$  with  $N(n, i)$  defined in (15). It is the normalization factor and is required for the operator valued  $SU(2)$  matrix to be unitary. For example, on a particular link (we suppress the link index  $(n, i)$ ),

$$\begin{aligned} (U^\dagger U)_{\alpha\beta} &= \frac{1}{\sqrt{N+1}} (a_\gamma b_\alpha + \tilde{a}_\gamma^\dagger \tilde{b}_\alpha^\dagger) \frac{1}{(N+1)} (a_\gamma^\dagger b_\beta^\dagger + \tilde{a}_\gamma \tilde{b}_\beta) \frac{1}{\sqrt{N+1}} \\ &= \frac{1}{\sqrt{N+1}} \left( a_\gamma b_\alpha \frac{1}{(N+1)} a_\gamma^\dagger b_\beta^\dagger + \tilde{a}_\gamma^\dagger \tilde{b}_\alpha^\dagger \frac{1}{(N+1)} \tilde{a}_\gamma \tilde{b}_\beta \right) \frac{1}{\sqrt{N+1}} \\ &= \delta_{\alpha,\beta}. \end{aligned} \quad (20)$$

The last step of equation (20) involves some simple algebra of harmonic oscillators and use of the constraint (15). Similarly, it can be checked that the operator valued matrix elements of  $U$  in (19) commute amongst themselves and (19) is consistent with the defining equation (4) for the generator of the right gauge transformations  $e(n, i)$ . We can also write  $U$  on a link in the form of a matrix:

$$U = \frac{1}{\sqrt{a^\dagger \cdot a + 1}} \begin{pmatrix} a_1^\dagger b_1^\dagger + a_2 b_2 & a_1^\dagger b_2^\dagger - a_2 b_1 \\ a_2^\dagger b_1^\dagger - a_1 b_2 & a_2^\dagger b_2^\dagger + a_1 b_1 \end{pmatrix} \frac{1}{\sqrt{a^\dagger \cdot a + 1}}.$$

This is the standard structure of a  $SU(2)$  matrix; i.e., it is of the form  $U = \begin{pmatrix} z_1 & -z_2^\dagger \\ z_2 & z_1^\dagger \end{pmatrix}$  with  $z_1^\dagger z_1 + z_2^\dagger z_2 = z_1 z_1^\dagger + z_2 z_2^\dagger = 1$  on  $\tilde{\mathcal{H}}$ . More rigorously, one can use the Wigner–Eckart theorem on  $\tilde{\mathcal{H}}$  to prove (19). For this, we expand [25]

$$U_{mm'}^j |J, M, M'\rangle = \sum_{K=|J-j|}^{J+j} \sqrt{\frac{2J+1}{2K+1}} \langle J, j, M, m | K, M+m \rangle \langle J, j, M', m' | K, M'+m' \rangle |K, M+m, M'+m'\rangle. \quad (21)$$

The symbols  $\langle J, j, M, m | K, M+m \rangle$  denote the Clebsch–Gordon coefficients. Using  $j = \frac{1}{2}$  and the values [25]

$$\begin{aligned} \left\langle J, \frac{1}{2}, M, \pm \frac{1}{2} \left| J - \frac{1}{2}, M \pm \frac{1}{2} \right. \right\rangle &= \mp \sqrt{\frac{J \mp M}{2J+1}}, \\ \left\langle J, \frac{1}{2}, M, \pm \frac{1}{2} \left| J + \frac{1}{2}, M \pm \frac{1}{2} \right. \right\rangle &= \sqrt{\frac{J \pm M + 1}{2J+1}} \end{aligned}$$

we recover the Hamiltonian in terms of pre-potentials (19). As an example,

$$\begin{aligned} U_{\frac{1}{2}\frac{1}{2}} |J, M, M'\rangle &= \sqrt{\frac{(J+M+1)(J+M'+1)}{(2J+1)(2J+2)}} \left| J + \frac{1}{2}, M + \frac{1}{2}, M' + \frac{1}{2} \right\rangle \\ &\quad + \sqrt{\frac{(J-M)(J-M')}{(2J)(2J+1)}} \left| J - \frac{1}{2}, M + \frac{1}{2}, M' + \frac{1}{2} \right\rangle \\ &= \sqrt{\frac{1}{a^\dagger \cdot a + 1}} (a_1^\dagger b_1^\dagger + a_2 b_2) \sqrt{\frac{1}{a^\dagger \cdot a + 1}} |J, M, M'\rangle. \end{aligned} \quad (22)$$

Similarly,  $U_{-\frac{1}{2}-\frac{1}{2}}, U_{\frac{1}{2}-\frac{1}{2}}, U_{-\frac{1}{2}\frac{1}{2}}$  can be explicitly worked out. Note that the simple  $SU(2) \otimes U(1)$  gauge transformation properties of the pre-potentials along with unitarity of the link operator had directly led us to this result. We see that the first and second terms in (21) or equivalently in (19) change the value of the angular momentum by  $+\frac{1}{2}$  and  $-\frac{1}{2}$  units respectively<sup>1</sup>. Thus we have broken the  $SU(2)$  link operators  $U(n, i)$  into the left ( $a_\alpha(n, i)$ ) and the right ( $b_\alpha(n, i)$ ) transforming pre-potentials. This separation will be crucial to construct the gauge invariant states of the theory in the next section. The Hamiltonian in (1) can now be written in the form

$$H = \sum_{n,i} \frac{N(n, i)}{2} \left( \frac{N(n, i)}{2} + 1 \right) + \sum_{\text{plaquettes}} \text{tr}(U_{\text{plaquette}} + U_{\text{plaquette}}^\dagger). \quad (23)$$

The first term in (23) depends on the number operator on all the links of the lattice. The second term is made up of the four links of the plaquettes given by (19). The Hamiltonian in (23) is trivially invariant under the  $SU(2) \otimes U(1)$  gauge transformations. We note that the two operators

$$U^{+\frac{1}{2}}(n, i)_{\alpha, \beta} \equiv a^\dagger(n, i)_\alpha b^\dagger(n, i)_\beta, \quad U^{-\frac{1}{2}}(n, i)_{\alpha, \beta} \equiv \tilde{a}(n, i)_\alpha \tilde{b}(n, i)_\beta \quad (24)$$

have the same gauge transformation properties as  $U(n, i)$  and they are both invariant under the  $U(1)$  gauge transformation. Therefore, we can define the gauge invariant operators as consisting of products of  $U^{\pm\frac{1}{2}}(n, i)$  over the links of a directed closed loop. For example, the simplest gauge invariant operators in terms of pre-potentials are defined over a plaquette

<sup>1</sup> In terms of the Young tableau, the first term corresponds to adding a new box in the horizontal row and the second term corresponds to deleting a box from the horizontal row.

$(n, i, j)$  and are given by

$$U^{\sigma_1}(n, i)U^{\sigma_2}(n + i, j)(U^{\sigma_3}(n + j, i))^\dagger(U^{\sigma_4}(n, j))^\dagger$$

where  $\sigma_1, \sigma_2, \sigma_3, \sigma_4$  are  $\pm\frac{1}{2}$ . Therefore, there are 16 such gauge invariant plaquette operators and any one of them can be used to define the magnetic term of the Hamiltonian. The magnetic term in the Hamiltonian (23) is simply the sum of all of them. We expect all such Hamiltonians to belong to the same universality class. The coupling of matter with the gauge fields is also simple in this formulation. Let  $(q_\alpha^\dagger(n), q_\alpha(n))$  be the doublets of matter creation–annihilation operators at site  $(n)$ . Under the  $SU(2)$  gauge transformations they transform as  $q_\alpha(n) \rightarrow V_{\alpha\beta}(n)q_\beta(n)$  and are singlets under  $U(1)$  gauge transformations. The  $SU(2) \otimes U(1)$  singlet interaction terms are  $(q^\dagger \cdot a^\dagger)(b^\dagger \cdot q)$  and  $(q^\dagger \cdot \tilde{a})(\tilde{b} \cdot q)$ . Note that the minimal coupling with the original link variable  $U_{\alpha\beta}$  is the sum of the above two operators. Again we expect that in the continuum limit all the choices will lead to the same physics.

It is interesting to understand the role of the Abelian gauge invariance in colour confinement [28, 29]. For this we imagine an external quark  $q(n)$  and an anti-quark  $\bar{q}(m)$  located at lattice sites  $n$  and  $m$  respectively. With pre-potentials, we can construct the  $SU(2)$  colour invariant states separately at  $(n)$  and  $(m)$  as  $(b^\dagger(n) \cdot q(n))$  and  $(q^\dagger(m) \cdot a^\dagger(m))$  respectively. Now to satisfy the  $U(1)$  gauge invariance, we need to introduce pre-potentials along one of the strings which connect the lattice sites  $n$  and  $m$ . Note that this string contains  $U(1)$  and not  $SU(2)$  flux and is required to get  $U(1)$  gauge invariant state. In the strong coupling limit, the energy of this string is proportional to its length leading to colour confinement. In the case of  $SU(N)$  lattice gauge theory, the string will carry  $U(1)^{N-1}$  fluxes. This is similar to the idea of 't Hooft [28] that in  $SU(N)$  gauge theory the  $U(1)^{N-1}$  group is relevant for colour confinement.

## 5. The physical Hilbert space $\tilde{\mathcal{H}}^P$

The Gauss law constraints project out the physical Hilbert space consisting of states belonging to the singlet representation of the gauge group. This construction of gauge invariant states has been extensively discussed both in the continuum and on the lattice [1, 2, 16–18, 19, 26]. There are two approaches to find the solutions of the Gauss law constraints. The first approach is by considering the set of all Wilson loops [1, 2, 16]. This approach provides a manifestly gauge invariant basis in terms of link operators. However, it is not orthonormal and is overcomplete. The overcompleteness arises from well-known Mandelstam identities [16, 19]. The second approach labels the physical Hilbert space by the Wigner  $D$  matrices. This approach provides a gauge invariant orthonormal basis but suffers from rapid proliferation of the Clebsch–Gordan coefficients and the  $SU(2)$  indices [11, 18]. We now show that the simple gauge transformation properties in the present formulation lead to a new way of solving the Gauss law constraints in terms of pre-potentials. The construction is in arbitrary  $d$  dimensions, manifestly gauge invariant and does not require Clebsch–Gordan coefficients. The physical Hilbert space is characterized by  $d(2d - 1)$  gauge invariant integers at every lattice site. In what follows, it is convenient to collect the set of  $2d$  pre-potential creation operators associated with the site  $(n)$  as  $c^\dagger(n, \vec{i})$  with  $\vec{i} = 1, 2, \dots, 2d$ , where

$$c^\dagger(n, i) \equiv \tilde{a}^\dagger(n, i) \quad c^\dagger(n, d + i) \equiv b^\dagger(n - i, i) \quad i = 1, 2, \dots, d.$$

With the above relabelling, we note the following simplifications:

- (1) The new pre-potentials also satisfy harmonic oscillator algebra:

$$[c_\alpha(n, \vec{i}), c_\beta^\dagger(n, \vec{j})] = \delta_{\alpha,\beta} \delta_{\vec{i}, \vec{j}}.$$



(2) The  $SU(2)$  Gauss law (9) at site  $n$  is

$$\begin{aligned} J_{\text{total}}^a(n) &\equiv \sum_{i=1}^{2d} c^\dagger(n, i)_\alpha \left( \frac{\sigma^a}{2} \right)_{\alpha\beta} c_\beta(n, i) \\ &\equiv \sum_{i=1}^{2d} J^a(ni) = 0 \end{aligned} \quad (25)$$

and it simply states that the sum of all the angular momenta meeting at the site  $(n)$ ,  $\vec{J}_{\text{total}}(n)$ , is zero.

(3) Under  $SU(2)$  gauge transformations, all the  $2d$  operators  $c^\dagger(n, \bar{i})$  transform from the right as  $SU(2)$  doublets.

Thus, the problem of constructing the most general  $SU(2)$  gauge invariant states at site  $n$  reduces to constructing  $SU(2)$  singlets out of  $2d$  spin half pre-potentials  $c_\alpha^\dagger(n\bar{i})$ . We denote the physical Hilbert space at site  $n$ , consisting of all such invariants by  $\tilde{\mathcal{H}}_n^p$ . Therefore, in terms of pre-potentials  $\tilde{\mathcal{H}}_n^p$  is characterized as

$$|\vec{l}(n)\rangle \equiv \left| \begin{array}{cccc} l_{12} & l_{13} & \cdots & l_{12d} \\ & l_{23} & \cdots & l_{22d} \\ & & \cdot & \cdot \\ & & & l_{2d-12d} \end{array} \right\rangle = \prod_{\substack{\bar{i}, \bar{j} \\ \bar{j} > \bar{i}}} (c^\dagger(n\bar{i}) \cdot \tilde{c}^\dagger(n\bar{j}))^{l_{\bar{i}\bar{j}}(n)} |0\rangle. \quad (26)$$

In (26),  $l_{ij}(n) (\equiv l_{ji}(n))$  are  $N_d = d(2d - 1)$  positive integers which are invariant under the  $SU(2)$  gauge transformations. Thus, in terms of the pre-potentials, the  $SU(2)$  Gauss law (25) is solved locally and all the solutions are of the form (26). The  $SU(2)$  gauge invariant Hilbert space can be trivially written as  $\prod_n \otimes \tilde{\mathcal{H}}_n^p$  where  $\otimes$  denotes the direct product. However, the gauge invariance in terms of pre-potentials is not just  $SU(2)$  but  $SU(2) \otimes U(1)$  and the physical states should also satisfy the  $U(1)$  Gauss law (15). For this purpose, we note that the states (26)  $\in \tilde{\mathcal{H}}_n^p$  are the eigenvectors of  $2d$  number operators or equivalently angular momentum operators:

$$c^\dagger(n\bar{i}) \cdot c(n\bar{i}) |\vec{l}(n)\rangle = \left( \sum_{\bar{j} \neq \bar{i}} l_{\bar{i}\bar{j}}(n) \right) |\vec{l}(n)\rangle, \quad \bar{i}, \bar{j} = 1, 2, \dots, 2d. \quad (27)$$

Therefore, in terms of the quantum numbers  $l_{ij}$  the  $U(1)$  Gauss law constraint (15) implies that

$$\sum_{\substack{j=1 \\ j \neq i}}^{2d} l_{ij}(n) = \sum_{\substack{j=1 \\ j \neq (d+i)}}^{2d} l_{(d+i)j}(n + \hat{i}) \quad \forall n, \quad i = 1, 2, \dots, d. \quad (28)$$

In (28),  $n$  and  $n + \hat{i}$  are the neighbouring sites in the  $i$ th direction. The complete  $SU(2) \otimes U(1)$  invariant Hilbert space can now be written as

$$\tilde{\mathcal{H}}^p = \prod_n' \otimes \tilde{\mathcal{H}}_n^p. \quad (29)$$

In (29), the direct product is taken over all the lattice sites. The  $'$  denotes that the vectors  $|\vec{l}(n)\rangle$  in  $\tilde{\mathcal{H}}_n^p$  at different lattice sites ( $n$ ) are chosen such that  $U(1)$  Gauss law (27) is satisfied. Thus we have explicitly constructed a manifestly  $SU(2)$  gauge invariant basis in terms of pre-potentials. This basis is overcomplete as there are too many  $(d(2d - 1))$  degrees of freedom per lattice site. The reason for this overcompleteness is that the basis is not an eigenstate of the complete set

of commuting operators. For simplicity, we illustrate this in  $d = 2$  dimensions. We now have four angular momenta meeting at any given site ( $n$ ). Before solving the  $SU(2) \otimes U(1)$  Gauss law, a complete orthogonal basis at  $n$  can be characterized by the eigenvalues of the complete set of eight commuting operators  $|J^2(i), J_z(i)\rangle, i = 1, 2, \dots, 4$  and  $J_z(i) = J^{a=3}(ni)$ . To solve the  $SU(2)$  Gauss law, we make a change of basis [17] and demand the following eight angular momentum operators to be diagonal:

$$|(J(n1))^2, (J(n2))^2, (J(n3))^2, (J(n4))^2, (J(n1) + J(n2))^2, (J(n3) + J(n4))^2, \\ (J(n1) + J(n2))_z, (J(n3) + J(n4))_z\rangle.$$

A further change of basis leads to

$$|(J(n1))^2, (J(n2))^2, (J(n3))^2, (J(n4))^2, (J(n1) + J(n2))^2, (J(n3) + J(n4))^2, \\ (J(n1) + J(n2) + J(n3) + J(n4))^2, (J(n1) + J(n2) + J(n3) + J(n4))_z\rangle.$$

According to the  $SU(2)$  Gauss law (25), the last two operators have zero eigenvalues and the eigenvalues of  $(J(n1) + J(n2))^2$  and  $(J(n3) + J(n4))^2$  are equal. Due to the  $U(1)$  Gauss law,  $(J(n3))^2 = (J(n-1, 1))^2$  and  $(J(n4))^2 = (J(n-2, 2))^2$ . Therefore, the  $SU(2) \otimes U(1)$  gauge invariant Hilbert space at a site ( $n$ ) can be uniquely characterized by the eigenvalues of [17]:

$$|(J(n1))^2, (J(n2))^2, (J(n1) + J(n2))^2 = (J(n3) + J(n4))^2\rangle. \quad (30)$$

The three quantum numbers in (30) correspond to the three physical degrees of freedom of the gluons at a site in  $d = 2$ . The above discussion can be easily generalized to higher dimensions leading to  $3(d-1)$  gauge invariant quantum numbers at each lattice site in the final state. We now see that the states in  $\mathcal{H}^p$ , with total angular momentum zero, are the eigenstates of only  $(J(n1))^2, (J(n2))^2$  but not of  $(J(n1) + J(n2))^2$ . Therefore, the basis given by (26) is overcomplete. However, the over completeness in (29) is not a serious problem as it can be removed by demanding that the states  $|\tilde{l}(n)\rangle$  also be the eigenstates of the remaining complete set of commuting operators. This will also make the basis orthogonal. Note that this procedure of finding an orthonormal gauge invariant basis is very different from solving Mandelstam constraints [19]. This construction of an orthonormal basis out of (26) in arbitrary  $d$  dimension is under progress and will be reported elsewhere. Some preliminary results are given in [30].

We now briefly discuss the pre-potential formulation for pure  $SU(N)$  lattice gauge theories. The rank of  $SU(N)$  group is  $(N-1)$  and therefore it has  $(N-1)$  fundamental representations. In terms of Young tableau, these representations consist of  $r$  ( $= 1, 2, \dots, N-1$ ) boxes arranged vertically and are of dimensions  $\frac{1}{r!}N(N-1)(N-2)\dots(N-r+1)$ . We denote the corresponding generators by  $\lambda^a[r]$  where  $= 1, 2, \dots, (N^2-1)$  and  $r = 1, 2, \dots, (N-1)$ . As all the operators are defined on links, we choose a particular link  $(n, i)$  and suppress this link index henceforth for convenience. We can now generalize (11) and use  $SU(N)$  generalization of  $SU(2)$  Schwinger boson algebra [15] to write

$$E^a = \sum_{r=1}^{N-1} a^\dagger[r] \frac{\tilde{\lambda}^a[r]}{2} a[r] \quad e^a = \sum_{r=1}^{N-1} b^\dagger[r] \frac{\lambda^a[r]}{2} b[r]. \quad (31)$$

The above defining relations for pre-potentials are invariant under

$$a^\dagger[r] \rightarrow \exp(i\theta[r])a^\dagger[r] \quad b^\dagger[r] \rightarrow \exp(i\phi[r])b^\dagger[r]. \quad (32)$$

In (32),  $\theta[r]$  and  $\phi[r]$  are  $2(N-1)$  arbitrary phase angles on the link  $(n, i)$  leading to additional  $U(1)^{N-1} \otimes U(1)^{N-1}$  Abelian gauge invariance. We note that the total occupation numbers  $a^\dagger[r] \cdot a[r]$  ( $b^\dagger[r] \cdot b[r]$ ) of the left (right) harmonic oscillators belonging to the  $r$ th

representation are the  $(N - 1)$  left (right)  $SU(N)$  Casimirs and gauge invariant. The  $SU(N)$  kinematical constraints [17] analogous to (15) are

$$a^\dagger[r] \cdot a[r] = b^\dagger[r] \cdot b[r], \quad r = 1, 2, \dots, N - 1. \quad (33)$$

The above constraints reduce the Abelian gauge invariance to  $U(1)^{N-1}$ . Thus the  $SU(N)$  lattice gauge theory in terms of pre-potentials has  $SU(N) \otimes U(1)^{N-1}$  gauge invariance. In terms of the above pre-potentials in the fundamental representations it would be interesting to construct the gauge invariant Hilbert space labelled by a set of integers at every site.

## 6. Discussion and Summary

In this work, we have reformulated lattice gauge theories in terms of operators which belong to the fundamental and not adjoint representation of the gauge group. These operators, called pre-potentials, are harmonic oscillators. The construction is completely based on symmetry arguments and is possible in any dimension. The Gauss law is solved in terms of pre-potentials and an explicit construction of a manifestly  $SU(2)$  gauge invariant basis is given. It will be interesting to construct the corresponding path integral formulation using  $SU(N)$  coherent states. In the case of  $SU(2)$  the formulation will involve  $SU(2)$  complex doublets on every link and will have  $SU(2) \otimes U(1)$  gauge invariance. The pre-potential approach may also be relevant to understand QCD in terms of colour singlet loop variables [27]. It would also be interesting to explore the connections with spin networks [31] which are useful in the context of quantum gravity. The work along these directions is in progress and will be reported elsewhere.

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